On the number of functions in a class of k-valued logic

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Abstract

Given an n-ary $k-$valued function f, $gap(f)$ denotes the essential arity gap of f. We obtain an explicit determination of n-ary $k-$valued functions f with $2 < gap(f) \leq n \leq k$. Our methods yield new combinatorial results about the number of $k-$valued functions with given gap.

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0 Introduction

Given a function f, the essential variables in f are defined as variables which occur in f and weigh with the values of that function. The number of essential variables is an important measure of complexity for discrete functions.

We shall obtain a few results concerning simplifying of functions by identification of variables.

The essential arity gap (gap) of Boolen functions are deeply investigated in [1, 2, 3].

In [4] R. Willard proved that if $n > k$ then $gap(f) \leq 2$.

1 Preliminaries

Let k be a natural number with $k > 2$ and let $K = \{0, 1, \ldots, k-1\}$ be the set (ring) of remainders modulo k. An *n-ary k-valued function (operation) on K* is a mapping $f : K^n \to K$ for a natural number n, called *the arity* of f. The set of the all such functions is denoted by P_k^n.

Definition 1.1 *Let $X_n = \{x_1, \ldots, x_n\}$ be the set of n variables. A variable x_i is called essential in f, or f essentially depends on x_i, if there exist values $a_1, \ldots, a_n, b \in K$, such that*

$$f(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n).$$

The set of all essential variables in a function f is denoted by $Ess(f)$ and the number of its essential variables is denoted by $ess(f) = |Ess(f)|$.

Let x_i and x_j be two distinct essential variables in f. We say that the function g is obtained from $f \in P_k^n$ by *the identification of the variable x_i with x_j*, if

$$g = f(x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_n) = f(x_i = x_j).$$

Briefly, when g is obtained from f, by identification of the variable x_i with x_j, we will write $g = f_{i \leftarrow j}$ and g is called *the identification minor of* f and $Min(f)$ denotes the set of all identification minors of f.

We shall allow formation of identification minors when x_i or x_j are not essential in f, also. Such minors of f are called *trivial* and they do not belong to $Min(f)$. So, if x_i does not occur in f, then we define $f_{i \leftarrow j} := f$.

Clearly, $ess(f_{i \leftarrow j}) \leq ess(f)$, because $x_i \notin Ess(f_{i \leftarrow j})$, even though it might be essential in f.

Definition 1.2 *Let $f \in P_k^n$ be an n-ary k-valued function. Then the essential arity gap (shortly arity gap or gap) of f is defined by*

$$gap(f) := ess(f) - \max_{g \in Min(f)} ess(g).$$

We let $G_{p,k}^m$ denote the set of all functions in P_k^n which essentially depend on m variables whose arity gap is p i.e. $G_{p,k}^m = \{f \in P_k^n \mid ess(f) = m \;\&\; gap(f) = p\}$, with $m \leq n$.

So, we shall consider the case $2 < k$ and $n \leq k$, solving a problem of M. Couceiro and E. Lehtonen, namely:
For each $1 \leq m \leq |A|$, determine explicitly the functions $f : A^n \to B$ whose arity gap is m ([1], page 6, Problem 1).

We shall assume that $A = B = K$. The most of the results obtained in this case might be easily generalized about finite defined and finite valued functions.

Let $m \in N$, $0 \leq m \leq k^n - 1$ be an integer. It is well known that for every $k, n \in N,\ k \geq 2$ there is an unique finite sequence $(\alpha_1, \ldots, \alpha_n) \in K^n$ such that

$$m = \alpha_1 k^{n-1} + \alpha_2 k^{n-2} + \ldots + \alpha_n.$$

The last equation is known as the representation of m in $k-$ary positional numerical system. One briefly writes $m = \overline{\alpha_1 \alpha_2 \ldots \alpha_n}$.

Given a variable x and $\alpha \in K$, x^α is an important function defined by:

$$x^\alpha = \begin{cases} 1 & if \quad x = \alpha \\ 0 & if \quad x \neq \alpha. \end{cases}$$

In this paper we shall use *sums of conjunctions (SC)* for representation of functions in P_k^n. This is the most natural representation of the functions in finite algebras. It is based on so called operation tables of the functions.

Each function $f \in P_k^n$ can be uniquely represented in SC-form as follows

$$f = a_0.x_1^0 \ldots x_n^0 \oplus \ldots \oplus a_m.x_1^{\alpha_1} \ldots x_n^{\alpha_n} \oplus \ldots \oplus a_{k^n-1}.x_1^{k-1} \ldots x_n^{k-1}$$

with $m = \overline{\alpha_1\alpha_2 \ldots \alpha_n}$, and $\alpha_i, a_m \in K$, where " $\oplus$ " and "." are the operations addition and multiplication modulo k in the ring K.

2 Essential Arity Gap of Functions

First, we study the n-ary k-valued functions whose arity gap is n.

Given two natural numbers $k, n \geq 2$, Eq_k^n denotes the set of all strings over $K = \{0, 1, \ldots, k-1\}$ with length n which have at least two equal letters i.e.

$$Eq_k^n := \{\alpha_1 \ldots \alpha_n \in K^n \mid \ \alpha_i = \alpha_j, \text{ for some } i, j \leq n, \ i \neq j\}.$$

Theorem 2.1 *Let $f \in P_k^n$, be a function which depends essentially on all of its n variables and $2 < n \leq k$. Then $f \in G_{n,k}^n$ if and only if it can be represented as follows*

$$f = [\bigoplus_{\beta_1 \ldots \beta_n \notin Eq_k^n} a_r.x_1^{\beta_1} \ldots x_n^{\beta_n}] \oplus a_0.[\bigoplus_{\alpha_1 \ldots \alpha_n \in Eq_k^n} x_1^{\alpha_1} \ldots x_n^{\alpha_n}], \tag{1}$$

where $r = \overline{\beta_1 \ldots \beta_n}$ and at least two among the coefficients $\{a_0\} \cup \{a_r \mid r = \overline{\beta_1 \ldots \beta_n}, \ \& \ \beta_1 \ldots \beta_n \notin Eq_k^n\}$, are distinct.

Corollary 2.1 *If $f \in G_{n,k}^n$ and $2 \leq n \leq k$, then $f(\alpha_1, \ldots, \alpha_n) = f(0, \ldots, 0)$ for all $\alpha_1 \ldots \alpha_n \in Eq_k^n$.*

Theorem 2.2 *If $2 \leq n \leq k$ then*

$$|G_{n,k}^n| = k^{\left(\binom{k}{n}.n!+1\right)} - k.$$

Lemma 2.1 *Let $f \in P_k^n$ be a k-valued function. If $x_i \notin Ess(f_{u \leftarrow v})$, $i, u, v \leq n$ and $i \notin \{u, v\}$ then $f_{u \leftarrow v} = g_{u \leftarrow v}$, where $g = f_{i \leftarrow j}$ for any arbitrary j, $j \in \{1 \ldots n\}$, $j \neq i$.*

Lemma 2.2 *Let $f \in P_k^n$ be a k-valued function. If $x_v \notin Ess(f_{u \leftarrow v})$ for some $u, v \leq n$, then $f_{u \leftarrow v} = g_{u \leftarrow j}$, where $g = f_{v \leftarrow j}$ for any arbitrary j, $j \in \{1 \ldots n\}$, $j \neq v$.*

Lemma 2.3 *Let $f \in P_k^n$ be a function depending essentially on all of its n variables and $1 \leq j < i \leq n$. If for each pair $(\alpha_i, \alpha_j) \in K^2$ with $\alpha_i \neq \alpha_j$, there exists $\gamma \in K$ such that $f(x_1, \ldots, x_{j-1}, \alpha_j, x_{j+1}, \ldots, x_{i-1}, \alpha_i, x_{i+1}, \ldots x_n) = \gamma$ then $X_n \setminus \{x_i, x_j\} \subseteq Ess(f_{i \leftarrow j})$.*

Theorem 2.3 *Let f be a k-valued function which depends essentially on the all of its n variables and $2 < p < n \leq k$. Then $f \in G^n_{p,k}$ if and only if there exist $n-p$ variables $y_{i_1}, \ldots, y_{i_{n-p}} \in X_n$ such that*

$$f(x_1, \ldots, x_n) = h(y_{i_1}, \ldots, y_{i_{n-p}}) \oplus g(x_1, \ldots, x_{n-p}, x_{n-p+1}, \ldots, x_n), \qquad (2)$$

where h depends essentially on all of its $n-p$ variables and $g \in G^n_{n,k}$ with $g_{i \leftarrow j} = 0$ for all $1 \leq i, j \leq n$.

Theorem 2.4 *If $2 < p < n \leq k$, then*

$$|G^n_{p,k}| = \left[k^{\left(\binom{k}{n}.n!\right)} - 1\right].\sum_{j=p}^{n}(-1)^{j-p}\binom{j}{p}\binom{n}{j}.k^{k^{n-j}}.$$

The functions depending essentially on three variables which have essential arity gap 2 are special in sense that almost all of the results proved for $p > 2$ or $n > 3$ are not satisfied here. In [4] it is shown that all functions with non-trivial (distinct from 1) arity gap belong to the class of functions $G^n_{2,k}$ when $n > k$. This class is deeply investigated and it is proved that it consists of totally symmetric functions.

The class $G^n_{2,k}$ when $n \leq k$ is quite interesting, also. We shall pay attention to this case, starting with description of the class $G^3_{2,3}$.

Example 2.1

To describe the functions from $G^3_{2,3}$ we need the following auxeliary functions.

$$s(x_1, x_2) := \bigoplus_{\beta=\alpha} x_1^\beta x_2^\alpha, \quad u^{(\alpha)}(x_1, x_2) := \bigoplus_{\beta\neq\alpha} x_1^\beta x_2^\beta, v^\alpha(x_1, x_2) := \bigoplus_{\beta\neq\alpha} x_1^\alpha x_2^\beta$$

Now, we might prove that $f \in G^3_{2,3}$, if and only if f can be represented in one of the following special forms:

$$f = \bigoplus_{i=0}^{2} a_0^{(i)}[x_3^i.s(x_1, x_2) \oplus x_2^i.u^{(i)}(x_1, x_3) \oplus x_1^i.u^{(i)}(x_2, x_3)] \oplus p_3(x_1, x_2, x_3),$$

$$f = \bigoplus_{i=0}^{2} a_0^{(i)}[x_1^i.x_2^i \oplus x_1^i.u^{(i)}(x_2, x_3) \oplus x_2^i.u^{(i)}(x_1, x_3)] \oplus p_3(x_1, x_2, x_3),$$

$$f = \bigoplus_{i=0}^{2} a_0^{(i)}[x_1^i.x_2^i \oplus x_2^i.v^{(i)}(x_3, x_1) \oplus x_2^i.u^{(i)}(x_1, x_3)] \oplus p_3(x_1, x_2, x_3),$$

$$f = \bigoplus_{i=0}^{2} a_0^{(i)}[x_1^i.x_2^i \oplus x_1^i.v^{(i)}(x_3, x_2) \oplus x_2^i.v^{(i)}(x_3, x_1)] \oplus p_3(x_1, x_2, x_3),$$

such that at least two among the coefficients $a_0^{(0)}, a_0^{(1)}, a_0^{(2)}$ *are different and* p_3 *are arbitrary functions defined by*

$$p_3(x_1, x_2, x_3) = \bigoplus_{\alpha_1\alpha_2\alpha_3 \notin Eq_3^3} a_m.x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}.$$

Proposition 2.1 $|G_{2,3}^3| = 192.729 = 139968.$

Without any difficulties, excluding the more complex calculations, we might generalize these results from $G_{2,3}^3$ to $G_{2,k}^3$ for arbitrary k, $k \geq 3$.

Proposition 2.2 $|G_{2,k}^3| = 8.729.\binom{k}{3}.(k^k - k) = 5832.\binom{k}{3}.(k^k - k).$

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